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Symmetrization Theorem of Full Steiner Trees

JIA FENG WENG

The Flinders University of South Australia,
Adelaide, South Australia 5042, Australia

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In this paper we prove the symmetrization theorem which says that the length of any full Steiner tree on a non-symmetric set A with a symmetric Steiner topology is equal to the length of the corresponding tree on a symmetric set. Hence, the Pollak theorem about the Steiner minimal tree on a quadrilateral becomes a simple corollary. The result can be extended to some symmetric topologies on six points.

1. SYMMETRIC FULL STEINER TOPOLOGY

A network T interconnecting a given set of points $A = \{a_1, a_2, ...\}$ with some additional points $s_1, s_2, ...$ is called a *Steiner tree* if it is a tree and satisfies the following conditions:

(i) Any two edges of T meet at an angle of at least 120°.

(ii) Any additional point is of degree at least 3.

To emphasize the dependence of T on A, sometimes we write T as $T(A)$. The given points a_i are referred to as *regular points* while the additional points s_i are referred to as *Steiner points*. If all regular points are of degree one, then T is called *full*. It is well known that the shortest network on A must be a Steiner tree and a Steiner tree can always be decomposed into a union of full Steiner subtrees. Hence, the full Steiner tree plays an essential role in the Steiner problem [3].

The graph structure of a network is called its *topology*. Hence, the topology of a Steiner tree (or full Steiner tree) is referred to as a *Steiner topology* (or *full Steiner topology*). Let t be a full Steiner topology whose points can be divided into two subsets of equal size $\{a_1, a_2, ..., s_1, s_2, ...\}$, $\{a'_1, a'_2, ..., s'_1, s'_2, ...\}$, where a_i, a'_i are regular points and s_i, s'_i are Steiner points. We call t *symmetric* if there is a one-to-one correspondence

$$f\colon a_i \leftrightarrow a'_i,\ s_i \leftrightarrow s'_i$$

preserving the adjacency and the direction of points; that is,

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(1) for any two points p and q, there is an edge linking p and q if and only if there is an edge linking p' and q';

(2) for any three points p, q, r adjacent to a same Steiner point s, the points p, q, r are counterclockwise if and only if p', q', r' are counterclockwise.

Clearly, f is also a one-to-one correspondence of edges, $f: e_k \leftrightarrow e'_k$, where $e_k = a_i s_j$ (or $= s_i s_j$) and $e'_k = a'_i s'_j$ (or $= s'_i s'_j$). For example, there are three possible symmetric full Steiner topologies on a set of six points, which are shown in Figs. 1(1)–1(3). Below we always use the superscript $'$ to represent the mapping f, i.e., the letters with $'$ are the images of the original elements.

Let (pq) denote the third vertex of the equilateral triangle on pq as the base which is on the left looking from p to q [1]. The length of pq is denoted by $|pq|$. (Similarly, the length of T is denoted by $|T|$.) Two *terminals*, i.e., points of degree 1, are called *associated terminals* if they are adjacent to a same Steiner point. By Melzak's construction [4], two associated terminals p, q in a Steiner tree T can be replaced by a new terminal (pq) or (qp), referred to as a *merging point*, to obtain a *reduced*

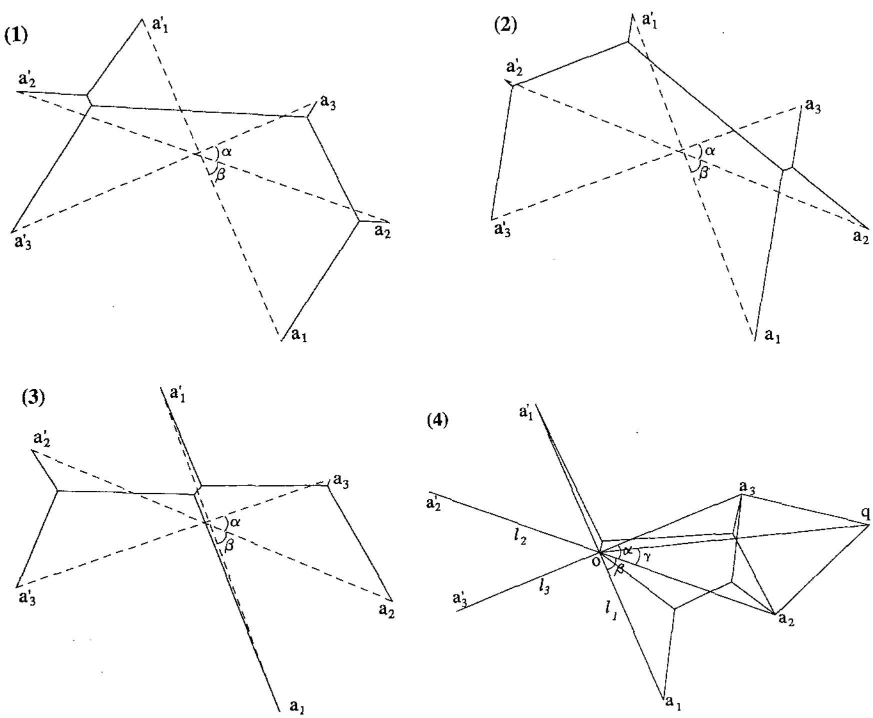

FIGURE 1.

Steiner tree T' so that $|T| = |T'|$. Hence, a full Steiner tree can be denoted by a sequence of parenthesized regular points after Cockayne [1]. For example, the full Steiner trees in Fig. 1(1) and Fig. 1(2) can be expressed by $T_1(A) = (a_3(a_2a_1))(a'_3(a'_2a'_1))$ and $T_2(A) = ((a_3a_2)a_1)((a'_3a'_2)a'_1)$, respectively. Because a Steiner tree is completely determined by its topology, the above notation is also applied to the topology of a full Steiner tree. So, the topology of the full Steiner tree in Fig. 1(1) is also expressed by $t_1 = (a_3(a_2a_1))(a'_3(a'_2a'_1))$. Suppose the full Steiner topology t is of the form

$$t = (a_{i_1}a_{i_2} \cdots a_{i_n})(a_{j_1}a_{j_2} \cdots a_{j_n})$$

after we remove all the parentheses except the outer two pairs. Then t is symmetric if and only if all corresponding subscripts are equal respectively, i.e., $i_k = j_k$ for $k = 1, 2, ..., n$.

2. Symmetrization Theorem of Full Steiner Trees

Suppose A is a set of $2n$ points $A = \{a_1, a_2, ..., a_n\} \cup \{a'_1, a'_2, ..., a'_n\}$. Suppose $b_ib'_i$ is obtained by translating $a_ia'_i$ so that all midpoints of $b_ib'_i$ coincide with each other. Then $B = \{b_1, b_2, ..., b_n\} \cup \{b'_1, b'_2, ..., b'_n\}$ is called a *symmetrized set* of A. Let the symmetric centre of B be o. Clearly, there are infinitely many symmetrized sets of A, but all of them are congruent. Let $l_i = a_ia'_i$, $i = 1, 2, ..., n$. Note that the symmetrization preserves the lengths of l_i and the angles between them. Suppose $T(A)$ is a full Steiner tree on A with a symmetric topology t and Steiner points $s_1, s_2, ..., s'_1, s'_2, ...$. Then we can construct a full Steiner tree $T(B)$ on B with the same topology t. Therefore, there is a natural one-to-one correspondence between the points of $T(A)$ and $T(B)$ as follows:

$$\begin{tikzcd}
\{a_i, s_i\} \arrow[r, leftrightarrow, "f"] \arrow[d, leftrightarrow, "g"] & \{a'_i, s'_i\} \arrow[d, leftrightarrow, "g"] \\
\{b_i, r_i\} \arrow[r, leftrightarrow, "f"] & \{b'_i, r'_i\}
\end{tikzcd}$$

where $b_1, b_2, ..., r_1, r_2, ...$ are the corresponding regular and Steiner points in $T(B)$. Of course, g is also a one-to-one correspondence between the edges of $T(A)$ and $T(B)$. Note that $g(e_k) = g(e'_k)$ for any edge e_k because B itself is a symmetric set.

Symmetrization Theorem. *Suppose A is a set of $2n$ points and t is a symmetric topology on A as stated in the above paragraph. Then*

(1) $|T(A)| = |T(B)|$.

(2) *Moreover,* $|e_k| + |e'_k| = |g(e_k)| + |g(e'_k)| = 2\,|g(e_k)|$ *for all edges* e_k *of* $T(A)$.

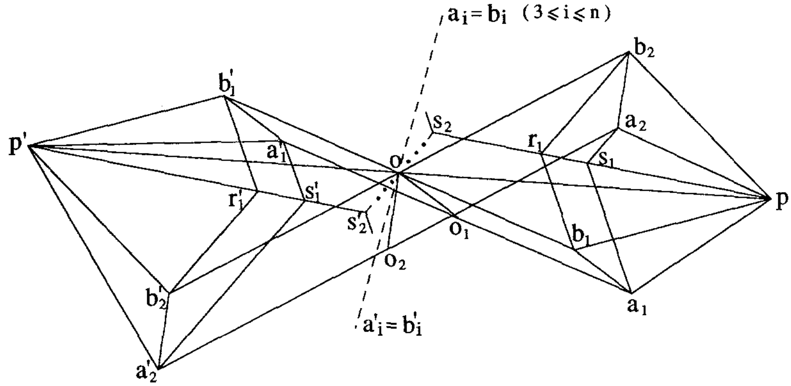

FIGURE 2.

Proof. We prove it by induction. Assume the theorem has been proved for $n-1$. Without loss of generality, suppose a_1 and a_2 are adjacent to the same Steiner point s_1. Let s_2 be the third point adjacent to s_1. Let $p=(a_2a_1)$ be the merging point. By the induction assumption, we may assume all of a_ia_i', $i=3, 4, ..., n$ and pp' have a common midpoint o, namely, $a_i=b_i$ for $i=3, 4, ..., n$ (Fig. 2).

By this assumption, o is also the midpoint of s_2s_2'. Let o_1, o_2 be the midpoints of a_1a_1', a_2a_2' respectively. Because Δa_2a_1p and $\Delta a_2'a_1'p'$ are directly similar (in fact, equilateral) and o_2, o_1, o divide a_2a_2', a_1a_1', pp' proportionately, by the theorem of similarity [2, p. 118], Δo_2o_1o is also an equilateral triangle. Suppose a_1a_1' is translated to b_1b_1' so that o_1 coincides with o. So is a_2a_2' translated to b_2b_2'. Note that $|pa_1|=|pa_2|$ and $|a_1b_1|=|o_1o|=|o_2o|=|a_2b_2|$. At the same time, $\angle b_1a_1p=\angle b_2a_2p$ because $|a_1b_1| \parallel |o_1o|$, $|a_2b_2| \parallel |o_2o|$ and $\angle a_1pa_2=\angle o_1oo_2$. Hence $\Delta pa_1b_1 \cong \Delta pa_2b_2$, $|pb_1|=|pb_2|$, $\angle b_1pb_2=\angle a_1pa_2=\pi/3$. This implies $p=(b_2b_1)$, and the Steiner point r_1, adjacent to b_2 and b_1, is on ps_2 as s_1. By symmetry, we have $p'=(b_2'b_1')$ and the Steiner point r_1', adjacent to b_2' and b_1', is on $p's_2'$. Hence $|T(A)|=|T(B)|$.

Now, let $e_1=a_1s_1$, $e_2=a_2s_1$, $e_3=s_1s_2$, $e_1'=a_1's_1'$, $e_2'=a_2's_1'$, $e_3'=s_1's_2'$, and let the other edges in $T(A)$ are e_k, e_k', $k=3, 4, ...$ By the induction assumption, we have $|e_k|+|e_k'|=|g(e_k)|+|g(e_k')|=2\,|g(e_k)|$, $k \geqslant 3$ already. It is easy to see that the angles of the quadrilateral $a_1b_1r_1s_1$ equal the corresponding angles of the quadrilateral $a_1'b_1'r_1's_1'$. Hence, $|s_1r_1|=|s_1'r_1'|$ since $a_1s_1 \parallel b_1r_1 \parallel a_1's_1' \parallel b_1'r_1'$ and $|a_1b_1|=|a_1'b_1'|$. It follows that

$$|e_3|+|e_3'|=|s_1s_2|+|s_1's_2'|=|r_1s_2|+|r_1's_2'|=|g(e_3)|+|g(e_3')|=2\,|g(e_3)|.$$

Because a_1a_1', the projection of the broken line $a_1s_1s_2s_2's_1'a_1'$, is equally as long as b_1b_1', the projection of the broken line $b_1r_1s_2s_2'r_1'b_1'$, and because the corresponding segments of the two broken lines are parallel, it is easily shown that

$$|e_1| + |e_1'| = |a_1 s_1| + |a_1' s_1'| = |b_1 r_1| + |b_1' r_1'| = |g(e_1)| + |g(e_1')| = 2\,|g(e_1)|.$$

Similarly, $|e_2| + |e_2'| = 2\,|g(e_2)|$.

Note that the above proof is also valid for $n=2$, i.e., when $s_2 = s_1'$, $s_2' = s_1$. So we complete the induction. ▌

Remark 1. Because $|g(e_i)| \geqslant \min\{|e_i|, |e_i'|\}$, hence, if $T(A)$ exists then $T(B)$ must exist.

Remark 2. In general, the length of a full Steiner tree T on a set of $2n$ points is a function of $4n-3$ independent variables [6]. But if the topology t is symmetric, then $|T|$ is the double of the length of the full subtree on the subset $\{b_1, b_2, ..., b_n, o\}$ by the symmetrization theorem. So, $|T|$ depends on only $2n-1$ variables; for example, on the length of l_i, $i = 1, 2, ..., n$, and $n-1$ angles between them.

3. Applications

There is a well-known result [5] about the full Steiner minimal tree on a quadrilateral $abcd$ (counterclockwise).

Pollak's Theorem. *Suppose both possible full Steiner trees on abcd exist. Then the shorter one is in the direction of the acute angle between the diagonals. They are equal if and only if the diagonals are perpendicular.*

Note that in the case of four points, both full Steiner topologies $(ad)(cb)$ and $(ba)(dc)$ are symmetric. Consequently, we may assume that $abcd$ is a parallelogram by the symmetrization theorem. Let the angle between the two diagonals, subtending ad, be θ. Clearly, the length of the two full Steiner trees are $\sqrt{|ac|^2 + |bd|^2 \pm |ac|\,|bd| \cos\theta}$. Therefore, Pollak's Theorem is a simple corollary of the symmetrization theorem.

Remark 3. Let the angle between ad and ac be α, the angle between ab and ac be β. Note that $\theta \leqslant \pi/2$ if and only if $|ad| \cos\alpha \leqslant |ab| \cos\beta$. Hence, Pollak's Theorem can be rewritten as

Theorem 1. *Suppose both full Steiner on abcd exist. Then* $|(ad)(cb)| \leqslant |(ba)(dc)|$ *if and only if* $|ad| \cos\alpha \leqslant |ab| \cos\beta$.

In this form, Pollak's Theorem can be extended to the case of six points. Let the angle between $a_3 a_3'$ and $a_2 a_2'$ be α, the angle between $a_1 a_1'$ and $a_2 a_2'$ be β as shown in Fig. 1. Then we have

THEOREM 2. *Suppose both full Steiner trees* $T_1(A)=(a_3(a_2a_1))(a_3'(a_2'a_1'))$ *and* $T_2(A)=((a_3a_2)\,a_1)((a_3'a_2')\,a_1')$ *exist. Then,* $|T_1(A)|\leqslant|T_2(A)|$ *if and only if* $|l_3|\cos\alpha\leqslant|l_1|\cos\beta$.

Proof. By the symmetrization theorem we may assume that a_1a_1', a_2a_2', and a_3a_3' have a common midpoint o. Hence the theorem follows from Remark 3 and Theorem 1 directly. ∎

Furthermore, we should note that the symmetrization of A involves only the translation of a_ia_i' but not the topology. Hence, it preserves the length of a full Steiner tree independent of the division of A into two symmetric subsets. For example, the full Steiner tree $T_3(A)=(a_1'(a_3a_2))(a_1(a_3'a_2'))$ in Fig. 1(3) is also symmetric, but A is divided into $\{a_2, a_3, a_1'\}\cup\{a_2', a_3', a_1\}$. In order to compare the lengths of T_2 and T_3, still by the symmetrization theorem, we may assume that a_1a_1', a_2a_2' and a_3a_3' have a common midpoint o. Let $q=(a_3a_2)$ and $\gamma=\angle qoa_2$ (Fig. 1(4)). Note that γ is fully determined by l_2, l_3 and the angle between them, denoted by α in Fig. 1(4). In fact,

$$|a_2q|^2=|a_2a_3|^2=|oa_2|^2+|oa_3|^2-2\,|oa_2|\,|oa_3|\cos\alpha,$$

$$|oq|^2=|a_2(oa_3)|^2=|oa_2|^2+|oa_3|^2-2\,|oa_2|\,|oa_3|\cos(\alpha+\pi/3),$$

$$|a_2q|^2=|oa_2|^2+|oq|^2-2\,|oa_2|\,|oq|\cos\gamma.$$

Hence,

$$\gamma=\gamma(l_2, l_3, \alpha)=\arccos\left(\frac{|l_2|+|l_3|(\sqrt{3}\sin\alpha+\cos\alpha)}{\sqrt{|l_2|^2+|l_3|^2-2\,|l_2|\,|l_3|\cos(\alpha+\pi/3)}}\right).$$

THEOREM 3. *Suppose both* $T_2(A)$ *and* $T_3(A)$ *exist. Then* $|T_2(A)|\leqslant|T_3(A)|$ *if and only if* $\gamma(l_2, l_3, \alpha)+\beta\leqslant\pi/2$.

Proof. By Remark 2, $|T_2(A)|=2\,|(a_3a_2)(a_1o)|$ and $|T_3(A)|=2\,|(a_3a_2)(oa_1')|$. Note that $|oa_1|=|oa_1'|$. Hence, the result follows from Pollak's Theorem directly. ∎

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